28 Introduction to the AdS/CFT Correspondence

28.1 D-branes

Strings can end on D-branes with Dirichlet boundary conditions. The in 10 dimensional type IIB string theory low-energy effective theory of N D3 branes is an $\mathcal{N} = 4$ $SU(N)$ gauge theory in $3 + 1$ dimensions, with a gauge coupling related to the string coupling by

$$g^2 = 4\pi g_s \ . \tag{28.1}$$

The metric around this setup is

$$\begin{aligned} ds^2 &= f^{\frac{-1}{2}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2) && (28.2)\\ f &= 1 + \left(\frac{R}{r}\right)^4 && (28.3)\\ R^4 &= 4\pi g_s \alpha'^2 N && (28.4) \end{aligned}$$

where r is the distance from the branes, and α' is the inverse string tension. Taking the low-energy limit corresponds to taking $r \to 0$ with

$$U = \frac{r}{\alpha'} \tag{28.5}$$

held finite. We find

$$\frac{ds^2}{\alpha'} = \frac{U^2}{\sqrt{4\pi g_s N}}(dt^2 + dx_i^2) + \sqrt{4\pi g_s N}(\frac{dU^2}{U^2} + d\Omega_5^2) \tag{28.6}$$

which is just the metric of five dimensional Anti-de Sitter Space (AdS_5) and a five sphere (S^5). Maldacena's conjecture is that Type IIB string theory on $AdS_5 \times S^5$ should be equivalent to $\mathcal{N} = 4$ $SU(N)$ gauge theory in $3 + 1$ dimensions. Supergravity is weakly coupled and hence a good approximation to type IIB string theory when

$$g_s \ll 1 \tag{28.7}$$

$$g_s N \gg 1 \ . \tag{28.8}$$

Perturbation theory is a good description of gauge theory when

$$g^2 \ll 1 \tag{28.9}$$

$$g^2 N \ll 1 \tag{28.10}$$

So this correspondence is hard to prove and potentially quite useful.

The sphere S^5:

$$R^2 = \sum_{i=1}^{6} Y_i^2 \tag{28.11}$$

has has a positive curvature and an $SO(6)$ isometry which corresponds to the $SU(4)_R$ symmetry of the gauge theory. AdS_5 can be embedded in 6 dimensions

$$ds^2 = -dX_0^2 - dX_5^2 + \sum_{i=1}^{4} dX_i^2 \tag{28.12}$$

as:

$$R^2 = X_0^2 + X_5^2 - (\sum_{i=1}^{4} X_i^2) \tag{28.13}$$

is a space with a negative curvature and a negative cosmological constant. We can change to "global" coordinates:

$$\begin{aligned}
X_0 &= R\cosh\rho\,\cos\tau && (28.14)\\
X_5 &= R\cosh\rho\,\sin\tau && (28.15)\\
X_i &= R\sinh\rho\,\Omega_i,\ i = 1,\ldots,4 && (28.16)\\
\sum_i \Omega_i^2 &= 1 && (28.17)
\end{aligned}$$

so

$$ds^2 = R^2(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega^2) \tag{28.18}$$

AdS_5 has an isometry $SO(2,4)$ which corresponds to the conformal symmetry group in 3+1 dimensions. Another coordinate choice is Poincare coordinates:

$$\begin{aligned}
X_0 &= \frac{1}{2u}\Big(1 + u^2(R^2 + \vec{x}^2 - t^2)\Big) && (28.19)\\
X_5 &= Rut && (28.20)\\
X_i &= Rux_i,\ i = 1,\ldots,3 && (28.21)\\
X_4 &= \frac{1}{2u}\Big(1 - u^2(R^2 - \vec{x}^2 + t^2)\Big) && (28.22)\\
& && (28.23)
\end{aligned}$$

so

$$ds^2 = R^2(\frac{du^2}{u^2} + +u^2(-dt^2 + d\vec{x}^2)) \tag{28.24}$$

which covers half of the space covered by the global coordinates. We can Wick rotate to a Euclidean version with

$$\tau \to \tau_E = -i\tau \tag{28.25}$$

$$\text{or}$$

$$t \to t_E = -it \tag{28.26}$$

with

$$\begin{aligned} ds_E^2 &= R^2(\cosh^2\rho d\tau_E^2 + d\rho^2 + \sinh^2\rho d\Omega^2) && (28.27) \\ &= R^2(\frac{du^2}{u^2} + +u^2(dt_E^2 + d\vec{x}^2)) && (28.28) \end{aligned}$$

which both cover the same space. Yet another coordinate choice is

$$u = \frac{1}{z} \tag{28.29}$$

$$x_4 = t_E \tag{28.30}$$

$$ds_E^2 = \frac{R^2}{z^2}(dz^2 + \sum_{i=1}^{4} dx_i^2) \tag{28.31}$$

The boundary of this space is R^4 at $z = 0$ and a point $z = \infty$. This boundary is the Wick rotation of M_4 and $u = 0$.

The a refined version of the conjectured correspondence is

$$\langle \exp \int d^4x \phi_0(x)\mathcal{O}(x)\rangle_{\text{CFT}} = Z_{\text{string}}\left[\phi(x,z)|_{z=0} = \phi_0(x)\right] \tag{28.32}$$

where we usually approximate

$$Z_{\text{string}} \approx e^{-I_{\text{sugra}}} \tag{28.33}$$

and $\mathcal{O}$ is an operator of the field theory while ϕ is a supergravity (or string) field in AdS_5.

References

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